On Multivariate Orthogonal Poynomials and elementary symmetric functions

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Abstract

We study families of multivariate orthogonal polynomials with respect to the symmetric weight function in d variables

$$B_\gamma(\mathtt{x}) = \prod_{i=1}^{d} w(x_i) \prod_{i<j} |x_i - x_j|^{2\gamma+1}, \quad \mathtt{x} \in (a,b)^d,$$

for $\gamma > -1$, where $w(t)$ is an univariate weight function in $t \in (a,b)$ and $\mathtt{x} = (x_1, x_2, \ldots, x_d)$ with $x_i \in (a,b)$. Using the change of variables $\mathtt{x} = (x_1, x_2, \ldots, x_d) \mapsto \mathtt{u} = (u_1, u_2, \ldots, u_d)$ where, u_r are the r**-th elementary symmetric functions** we study multivariate orthogonal polynomials in the variable $\mathtt{u}$ associated with the weight function $W_\gamma(\mathtt{u})$ defined by means of $W_\gamma(\mathtt{u}) = B_\gamma(\mathtt{x})$. For the new weight function, the domain is described in terms of the discriminant of the polynomial having x_i, $i = 1, 2, \ldots, d$, as its zeros and in terms of the associated Sturm sequence. Obviously, generalized classical orthogonal polynomials as defined by Lassalle [2, 3, 4] and Macdonald [5] are included in our study. Choosing the univariate weight function as the Hermite, Laguerre and Jacobi weight functions, we obtain the representation in terms of the variables u_r for the partial differential operators having the respective Hermite, Laguerre and Jacobi generalized multivariate orthogonal polynomials as the corresponding eigenfunctions. The case $d = 2$ coincides with the polynomials studied by Koornwinder in [1]. Finally, we explicitly present the partial differential operators for Hermite, Laguerre and Jacobi generalized polynomials in the cases $d = 2$ and $d = 3$.

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References

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